Fourier Transform and Applications

Fourier

Overview

- Transforms
  - Mathematical Introduction
- Fourier Transform
  - Time-Space Domain and Frequency Domain
  - Discret Fourier Transform
    - Fast Fourier Transform
  - Applications
- Summary
- References

Transforms

- Transform:
  - In mathematics, a function that results when a given function is multiplied by a so-called kernel function, and the product is integrated between suitable limits. (Britannica)

- Can be thought of as a substitution

$$F(s) = \{\mathcal{L}f\}(s) = \int_{0^-}^{\infty} e^{-st} f(t)\, dt.$$

Transforms

- Example of a substitution:
- Original equation: $x_4 + 4x^2 - 8 = 0$
- Familiar form: $ax^2 + bx + c = 0$
- Let: $y = x^2$
- Solve for y
- $x = \pm\sqrt{y}$

Transforms

- Transforms are used in mathematics to solve differential equations:
  - Original equation
  - Apply Laplace Transform
  - Take inverse Transform: $y = L^{-1}(y)$

Fourier Transform

- Property of transforms:
  - They convert a function from one domain to another with no loss of information
- Fourier Transform:

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

converts a function from the time (or spatial) domain to the frequency domain

Linearity of the Fourier Transform

If $x(t) \overset{F}{\leftrightarrow} X(j\omega)$

and $y(t) \overset{F}{\leftrightarrow} Y(j\omega)$

Then $ax(t) + by(t) \overset{F}{\leftrightarrow} aX(j\omega) + bY(j\omega)$

This follows directly from the definition of the Fourier transform (as the integral operator is linear). It is easily extended to a linear combination of an arbitrary number of signals

Time Shifting

- If $x(t) \overset{F}{\leftrightarrow} X(j\omega)$
- Then $x(t-t_0) \overset{F}{\leftrightarrow} e^{-j\omega t_0} X(j\omega)$
- Proof $$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$
- Now replacing t by $t-t_0$ $$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega$$ $$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0} X(j\omega)\right)e^{j\omega t}d\omega$$
- Recognising this as $$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega)$$
- A signal which is shifted in time does not have its Fourier transform magnitude altered, only a shift in phase.

Example: Linearity & Time Shift

- Consider the signal (linear sum of two time shifted steps)

$$x(t) = 0.5x_1(t-2.5) + x_2(t-2.5)$$

where $x_1(t)$ is of width 1, $x_2(t)$ is of width 3, centred on zero.

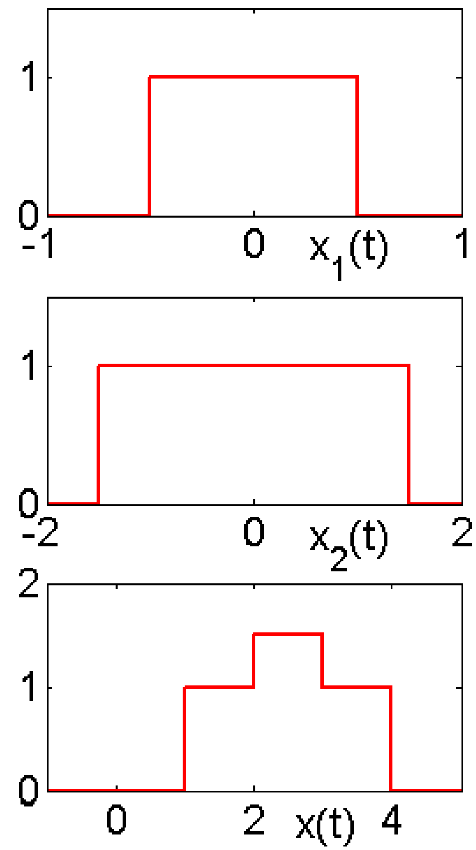
Using the rectangular pulse example

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega}$$

$$X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}$$

Then using the **linearity** and **time shift** Fourier transform properties

$$X(j\omega) = e^{-j5\omega/2}\left(\frac{\left(\sin(\omega/2) + 2\sin(3\omega/2)\right)}{\omega}\right)$$



Differentiation & Integration

By differentiating both sides of the Fourier transform synthesis equation:

Therefore: $$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)e^{j\omega t}d\omega$$

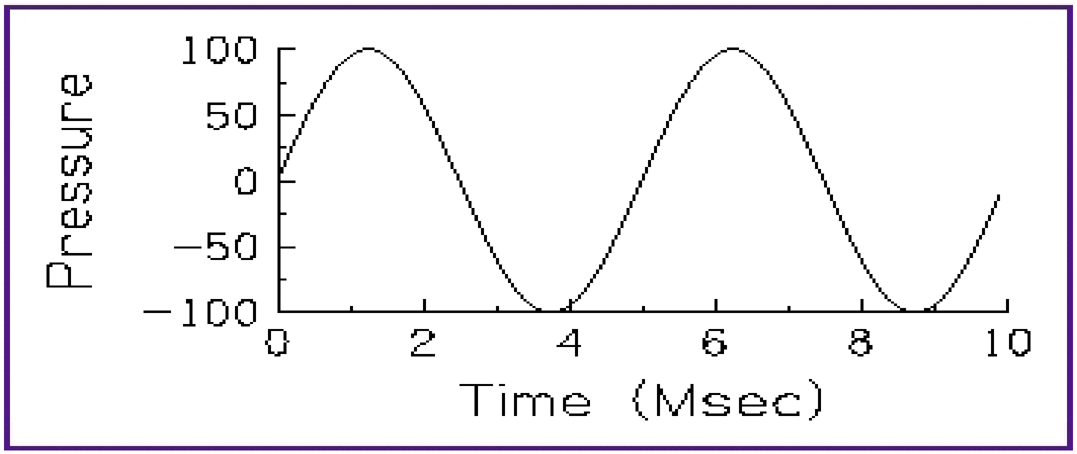
This is important, because it replaces **differentiation** in the **time domain** with **multiplication** in the **frequency domain**.

Integration is similar: $$\frac{dx(t)}{dt} \overset{F}{\leftrightarrow} j\omega X(j\omega)$$

The impulse term represents the dc or average value that can result from integration $$\int_{-\infty}^{t} x(\tau)d\tau = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Time Domain and Frequency Domain

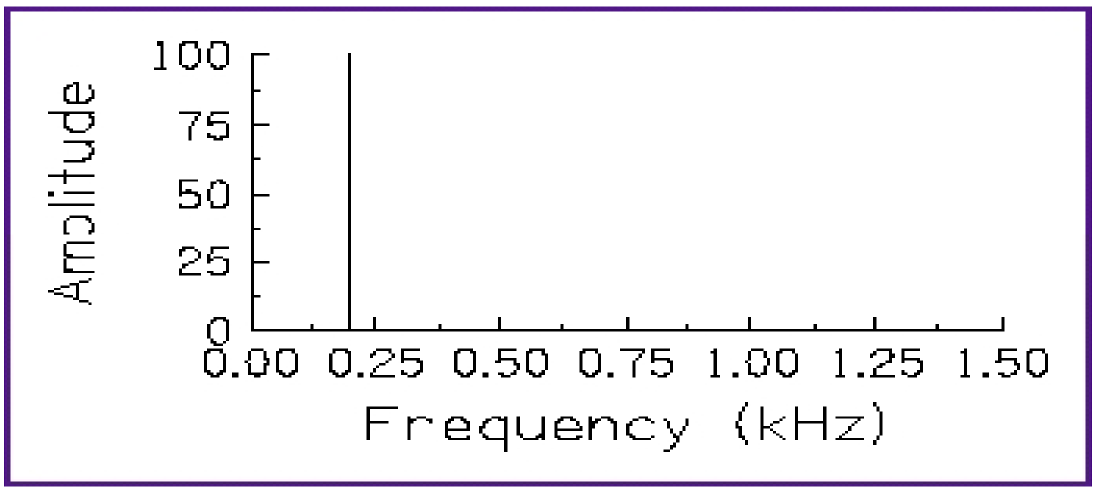
- Time Domain:
  - Tells us how properties (air pressure in a sound function, for example) change over time:



- Ampl
- Frequency = number of cycles in one second = 200 Hz

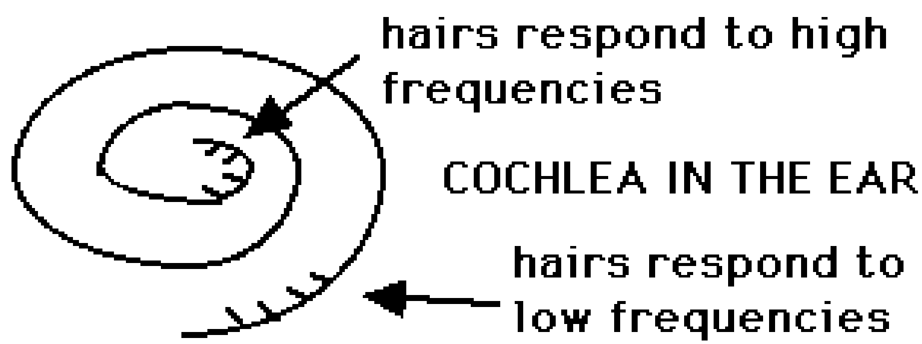
Time Domain and Frequency Domain

- Frequency domain:
  - Tells us how properties (amplitudes) change over frequencies:



Time Domain and Frequency Domain

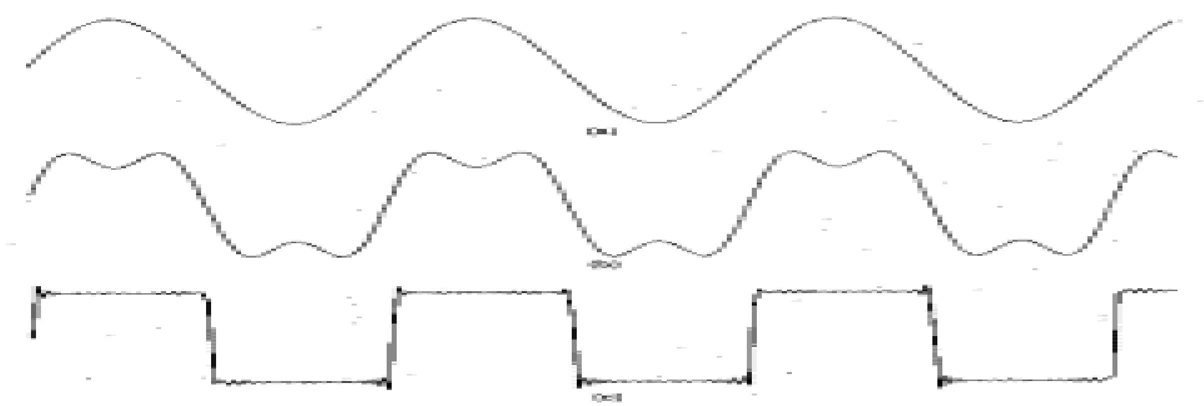
- Example:
  - Human ears do not hear wave-like oscilations, but constant tone



- Often it is easier to work in the frequency domain

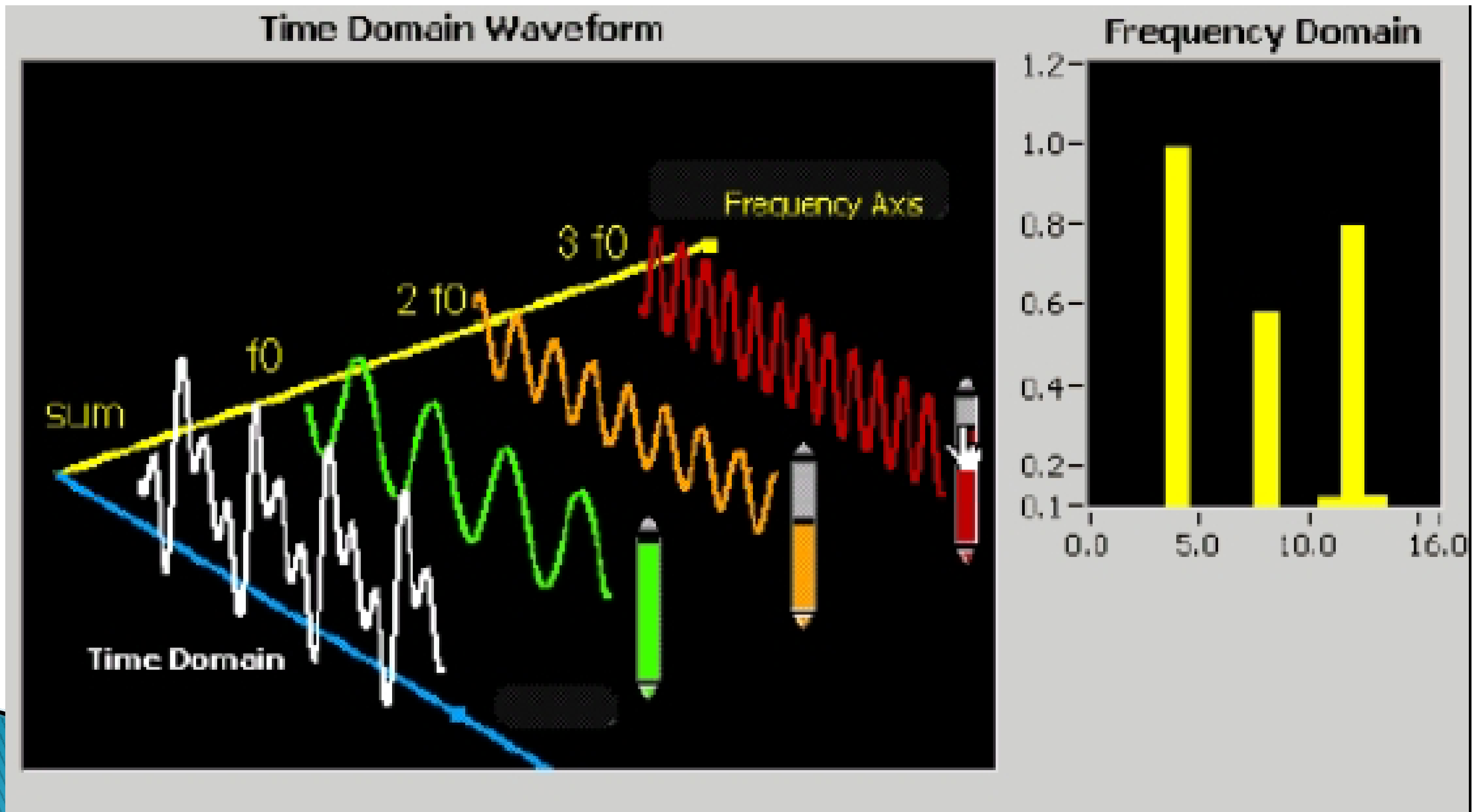
Time Domain and Frequency Domain

- In 1807, Jean Baptiste Joseph Fourier showed that any periodic signal could be represented by a series of sinusoidal functions



In picture: the composition of the first two functions gives the bottom one

Time Domain and Frequency Domain



Fourier Transform

- Because of the property:

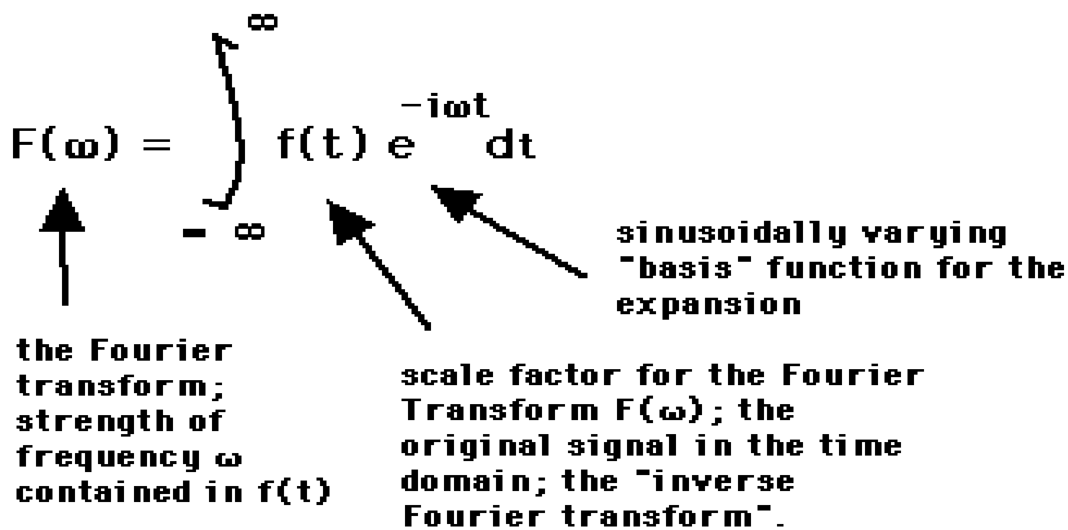
EULER's FORMULA

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{i\omega t} = \cos\omega t + i\sin\omega t$$

$$\text{where } i = \sqrt{-1}$$

- Fourier Transform takes us to the frequency domain:



Discrete Fourier Transform

- In practice, we often deal with discrete functions (digital signals, for example)
- Discrete version of the Fourier Transform is much more useful in computer science:

$$f_j = \sum_{k=0}^{n-1} x_k e^{-\frac{2\pi i}{n} jk} \qquad j = 0, \ldots, n-1$$

- $O(n^2)$ time complexity

Fast Fourier Transform

- Many techniques introduced that reduce computing time to O(n log n)
- Most popular one: **radix-2** decimation-in-time (**DIT**) FFT Cooley-Tukey algorithm:

$$
\begin{aligned}
f_j &= \sum_{k=0}^{\frac{n}{2}-1} x_{2k} e^{-\frac{2\pi i}{n} j(2k)} + \sum_{k=0}^{\frac{n}{2}-1} x_{2k+1} e^{-\frac{2\pi i}{n} j(2k+1)} \\
&= \sum_{k=0}^{n'-1} x'_k e^{-\frac{2\pi i}{n'} jk} + e^{-\frac{2\pi i}{n} j} \sum_{k=0}^{n'-1} x''_k e^{-\frac{2\pi i}{n'} jk} \\
&= \begin{cases} f'_j + e^{-\frac{2\pi i}{n} j} f''_j & \text{if } j < n' \\ f'_{j-n'} - e^{-\frac{2\pi i}{n}(j-n')} f''_{j-n'} & \text{if } j \geq n' \end{cases}
\end{aligned}
$$

(Divide and conquer)

Applications

- In image processing:
  - Instead of time domain: *spatial domain* (normal image space)
  - *frequency domain:* space in which each image value at image position F represents the amount that the intensity values in image I vary over a specific distance related to F

Example 1: Solving a First Order ODE

Calculate the response of a CT LTI system with impulse response:

$$h(t) = e^{-bt}u(t) \qquad b > 0$$

to the input signal:

$$x(t) = e^{-at}u(t) \qquad a > 0$$

Taking Fourier transforms of both signals:

$$H(j\omega) = \frac{1}{b + j\omega}, \quad X(j\omega) = \frac{1}{a + j\omega}$$

gives the overall frequency response:

$$Y(j\omega) = \frac{1}{(b + j\omega)(a + j\omega)}$$

to convert this to the time domain, express as **partial fractions**:

$$Y(j\omega) = \frac{1}{b - a}\left(\frac{1}{(a + j\omega)} - \frac{1}{(b + j\omega)}\right)$$

assume $b \neq a$

Therefore, the CT system response is:

$$y(t) = \tfrac{1}{b-a}\left(e^{-at}u(t) - e^{-bt}u(t)\right)$$

Example 2: Design a Low Pass Filter

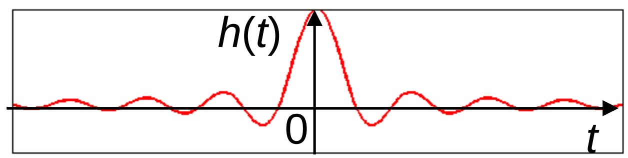
Consider an ideal **low pass filter** in frequency domain:

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

$$Y(j\omega) = \begin{cases} X(j\omega) & |\omega| < \omega_c \\ 0 & |\omega| > \omega_c \end{cases}$$

The **filter's impulse response** is the **inverse Fourier transform**

$$h(t) = \tfrac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega t}\, d\omega = \frac{\sin(\omega_c t)}{\pi t}$$



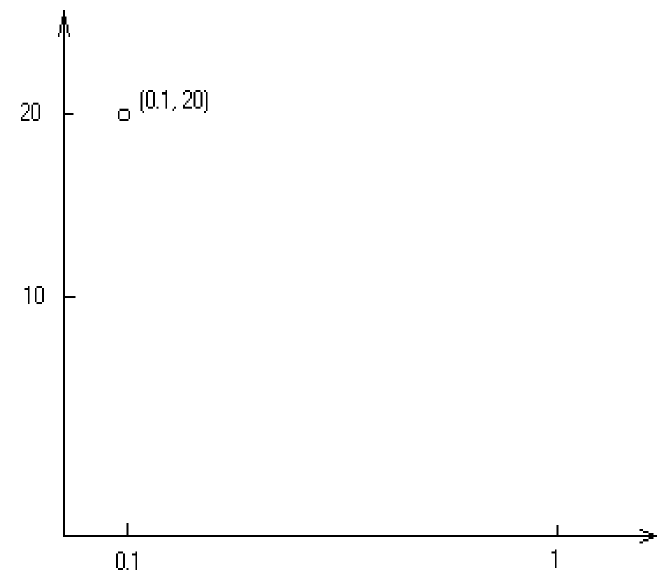
which is an ideal low pass CT filter. However it is non-causal, so this cannot be manufactured exactly & the time-domain oscillations may be undesirable

We need to approximate this filter with a causal system such as 1st order LTI system impulse response $\{h(t), H(j\omega)\}$:

$$a^{-1}\frac{\partial y(t)}{\partial t} + y(t) = x(t), \qquad e^{-at}u(t) \overset{F}{\leftrightarrow} \frac{1}{a + j\omega}$$

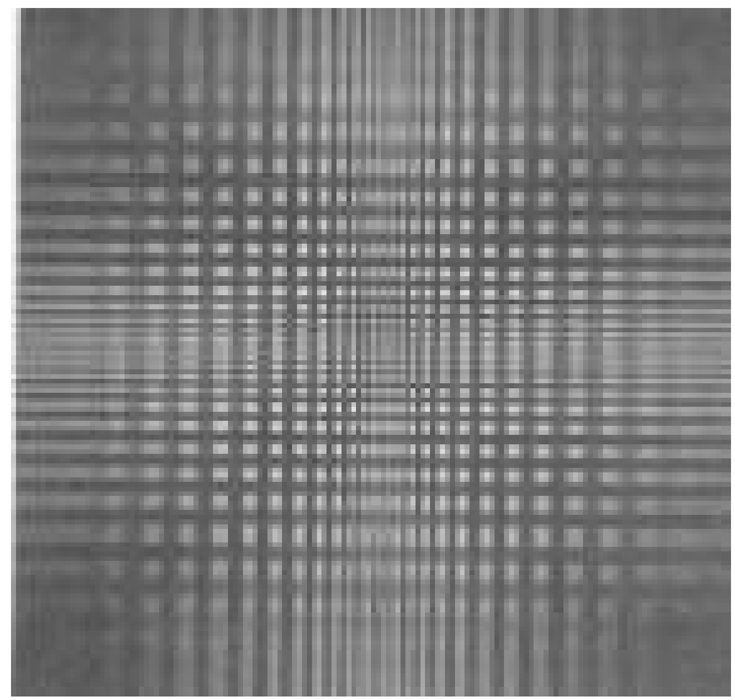
Applications: Frequency Domain In Images

- If there is value 20 at the point that represents the frequency 0.1 (or 1 period every 10 pixels). This means that in the corresponding spatial domain image I the intensity values vary from dark to light and back to dark over a distance of 10 pixels, and that the contrast between the lightest and darkest is 40 gray levels



Applications: Frequency Domain In Images

- *Spatial frequency* of an image refers to the rate at which the pixel intensities change
- In picture on right:
  - High frequences:
    - Near center
  - Low frequences:
    - Corners



Applications: Image Filtering

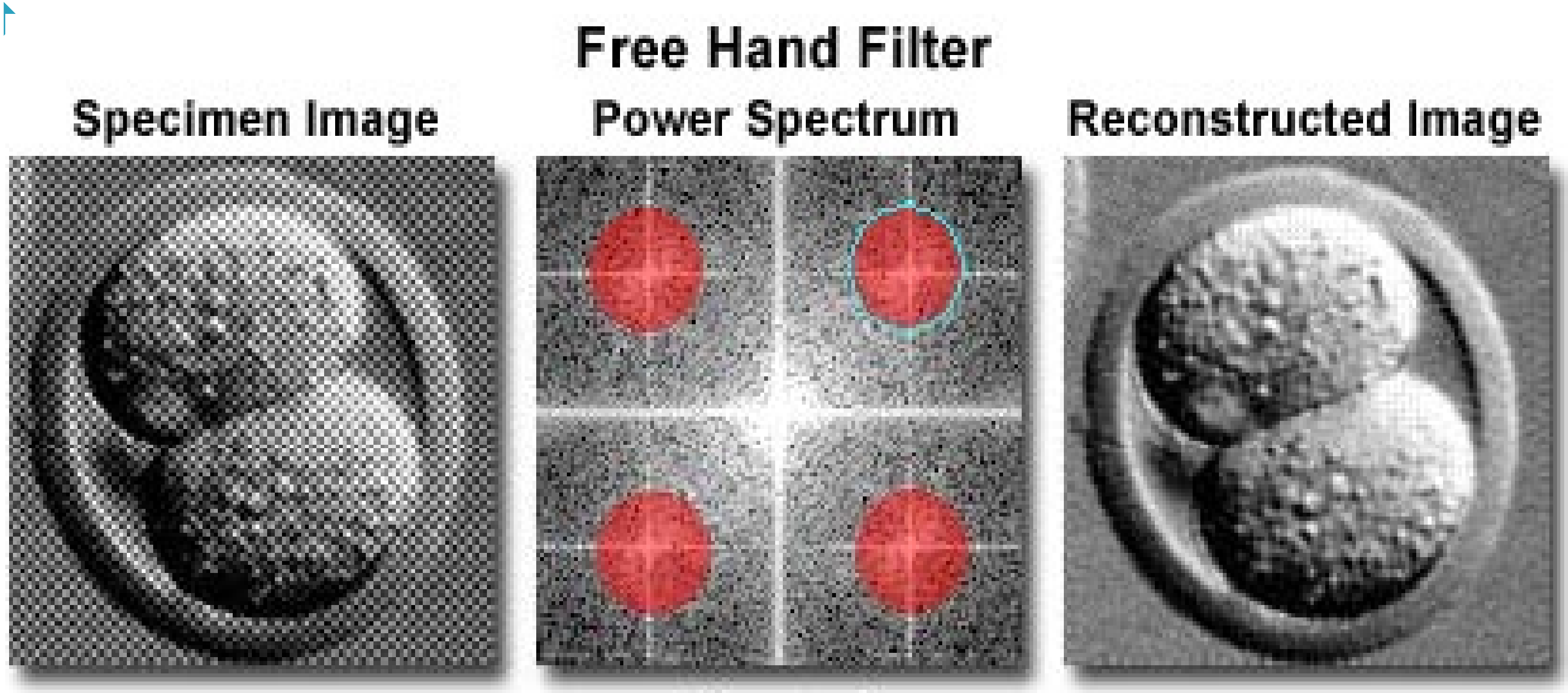


Figure 1

Other Applications of the DFT

- Signal analysis
- Sound filtering
- Data compression
- Partial differential equations
- Multiplication of large integers

Summary

- Transforms:
  - Useful in mathematics (solving DE)
- Fourier Transform:
  - Lets us easily switch between time-space domain and frequency domain so applicable in many other areas
  - Easy to pick out frequencies
  - Many applications

Assignment-1

- Try yourself

Q1. Find the Fourier sine transform of $\dfrac{e^{-ax}}{x}$

Q2. Find the Fourier sine transform of e^{-ax}. Hence evaluate $\displaystyle\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx$

Q3. Find the Fourier sine transform of $\dfrac{1}{x}$

Q4. Find the Fourier cosine transform of e^{x^2}

Q5. Find the Fourier cosine transform of $f(x)=\begin{cases}\cos x & , 0<x<a \\ 0 & , x>a\end{cases}$

Assignment-2

- Try yourself

Q1. Using Parseval's $\int_0^\infty \frac{t^2}{(4+t^2)(9+t^2)}dt = \frac{\pi}{10}$

Q2. Find the Fourier cosine transform of $f(x) = \begin{cases} 1-|x| & , |x|<1 \\ 0 & , |x|>1 \end{cases}$

Q3. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x>0, \quad t>0$

subject to the condition

$(i).\quad u=0, \quad when \quad x=0, \qquad t>0$

$(ii).\ u = \begin{cases} 1, & 0<x<1 \\ 0, & x \geq 1 \end{cases} \qquad when\ t=0$

and

$(iii).\ u(x,t)$ *is bounded*.

Assignment-3

- Try yourself

Q1. Use finite Fourier transform, solve $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$

subject to the condition

$$(i).\quad u_x(0,\ t) = u_x(6,\ t) = 0, \qquad 0 < x < 6, \qquad t > 0$$

$$(ii).\ u\ (x,\ 0) = x(6 - x) = 0, \qquad 0 < x < 6$$

Q2. Verify convolution theorem for $F(x) = G(x) = e^{-x^2}$

- Thank you